Thermodynamics of Local Causal Horizons

Arif Mohd*
SISSA - International School for Advanced Studies,
Via Bonomea 265, 34136 Trieste, Italy
and
INFN, Sezione di Trieste

Sudipta Sarkar†
Indian Institute of Technology, Gandhinagar, India
(Dated: April 9, 2013)

We propose an expression for the entropy density associated with the Local Causal Horizons in any diffeomorphism invariant theory of gravity. If the black-hole entropy of the theory satisfies the physical process version of the first law of thermodynamics then our proposed entropy satisfies the Clausius relation. Thus, our study shows that the thermodynamic nature of the spacetime horizons is not restricted to the black holes but it also applies to the local causal horizons in the neighborhood of any point in the spacetime.

Black-hole physics has provided strong hints of a deep and fundamental relationship between gravitation, thermodynamics and quantum theory. At the heart of this relationship is black-hole thermodynamics, which says that certain laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole [1]. Classical and semi-classical analyses of the thermodynamic behavior of black holes has given rise to most of our present physical insights into the nature of quantum phenomena occurring in strong gravitational fields. For example, the holographic principle emerged through various thought experiments related to the sub-extensive scaling of Bekenstein-Hawking entropy - and applied it to arbitrary gravitational systems.

The equilibrium state version of the first law of black-hole thermodynamics for arbitrary diffeomorphism-invariant theory of gravity was established by Wald and collaborators [2, 3]. The entropy of the black hole can be expressed as a local geometric quantity integrated over a spacelike cross section of the horizon and is identified with the Noether charge corresponding to the Killing isometry that generates the horizon. It is also possible to write down a quasi-stationary version of the second law for Lanczos-Lovelock gravity [4, 5] for physical processes in which the horizon is perturbed by the accretion of positive energy matter and the black hole ultimately settles down to a stationary state.

The discovery of the black-hole thermodynamics was also a precursor to the idea of emergent gravity, which asserts that spacetime is an emergent notion, a macroscopic and coarse-grained description of the underlying degrees of freedom which are fundamentally described by a quantum theory of gravity. But how could gravitation be an emergent phenomenon if the thermodynamics is tied to special spacetimes - those containing a black hole? In this direction a profound insight was provided by Jacobson who managed to derive the Einstein equation as a consequence of the thermodynamics associated to the Local Causal Horizons (LCHs) [6]. Jacobson showed that if one identifies a multiple of area with the entropy ($S \propto A$) associated with a LCH at boost temperature T, and if the change in the entropy δS is related to the flow of the boost energy δQ across the local horizon in such a way that the Clausius relation $\delta S = \delta Q/T$ is satisfied, then it is possible to derive the Einstein equation. The Einstein equation can therefore be interpreted as a thermodynamic equation of state.

This simple and elegant result has far reaching consequences. It supports the idea that the dynamics of space time can be interpreted as a thermodynamic description of some underlying degrees of freedom. Also, as pointed out by Padmanabhan [7], one can deduce the presence of these underlying degrees of freedom by the very observation that one can heat the system. The existence of physical quantities which provide a thermodynamical interpretation of the geometry of spacetime and the existence of a thermodynamical law (the Clausius relation) that gives rise to the dynamics of the geometry (the Einstein equation) is highly suggestive of the idea that the dynamical geometry itself could be an emergent notion borne out of the dynamics of some underlying degrees of freedom.

On very general grounds, however, we know that Einstein equation is not the end of the story. The Einstein-Hilbert action has just the first of the terms in a derivative expansion and one expects *à la* Wilson that higher-order curvature terms respecting the symmetries should also appear in the effective action. It is natural to ask if the thermodynamical nature of LCHs is unique to general relativity or is it a robust feature of any diffeomorphism invariant theory of gravity. Although there have been many attempts [8–13] to answer this question and extend the result of Ref. [6] to other theories of gravity, none of these are completely satisfactory. Either they work only for a very specific class of theories or they require extra

*Electronic address: arif.mohd@sissa.it
†Electronic address: sudiptas@iitgn.ac.in

assumptions. A detailed critique and the limitations of these attempts is discussed in Ref. [13].

In this paper we take the point of view that if the thermodynamics of LCHs is a fundamental property of a theory of gravity and there are internal degrees of freedom whose coarse-grained description is provided by the low energy gravitational effective action then we must be able to show the existence of thermodynamical laws associated to the LCHs with appropriately defined quantities as the thermodynamic variables. Hence, similar to black hole thermodynamics, the LCH must obey various thermodynamical laws. Thus, our goal is opposite to the one that Jacobson had in Ref. [6]. Instead of deriving the field equation of gravity from the assumed thermodynamics associated to the LCHs, we use the field equations to derive the Clausius relation for the LCHs. More precisely, for any given diffeomorphism invariant theory of gravity in which black holes satisfy the physical process version of the first law of thermodynamics, we propose an expression of the entropy associated with the LCHs that satisfies the Clausius relation. In other words, we generalize the thermodynamics of LCH from general relativity to any arbitrary diffeomorphism invariant theory of gravity.

Geometric setup.— We start with the construction of the LCH based at any point p in spacetime. We first choose a $(D-2)$-dim spacelike surface Σ containing p and we choose one side of the boundary of the past of Σ. In a small neighborhood of p this boundary is generated by a congruence of null geodesics k^a normal to Σ. Let λ be the affine parameter along the integral curves of k^a such that $\lambda = \lambda_f$ at p. Our choice of the patch Σ is such that the expansion θ and the shear σ_{ab} of k^a vanish at p. Our chosen side of the null boundary of the past of such a Σ is called the local causal horizon (LCH)[1] at p, see Fig. 1.

In order to define the heat flux and the temperature we need an approximate boost Killing vector field. Consider a point p_0 in the past of p lying at the value of the affine parameter $\lambda = 0$ on the central generator (i.e., the generator that passes through p). Next we construct an approximate boost Killing vector field ξ^a such that the point p_0 serves as a bifurcation point, i.e., ξ^a vanishes at p_0. On the central generator the approximate boost Killing vector is related to the null-geodesic generator as $\xi^a = \lambda k^a$. Furthermore, ξ^a satisfies the Killing equation $\nabla_{(a}\xi_{b)} = 0$ exactly at p, and to $\mathcal{O}(x)$ near p. We suggest the interested reader to consult the references [13, 14] for further details pertinent to this construction. The appropriate temperature is $T = 1/2\pi$ which is the boost temperature associated with the approximate Killing vector ξ^a. The heat flux is given in terms of the matter stress tensor and the approximate boost Killing vector field ξ^a as

$$\delta Q = \int_0^{\lambda_f} d\lambda\, dA\, \sqrt{\gamma}\; T_{ab}\xi^a k^b = \int_0^{\lambda_f} \lambda\, d\lambda\, dA\, \sqrt{\gamma}\; T_{ab}k^a k^b. \tag{1}$$

Here γ is the induced metric on the LCH and the dA integral is over a thin pencil of generators around the central generator connecting p to p_0.

We also need an entropy associated with the LCH. To begin with, taking a clue from the black-hole thermodynamics, it seems reasonable to guess that one should associate the same expression of entropy to LCHs as that of the black holes in the theory. Let us then assume that the underlying theory of gravity obeys an equation $E_{ab} = 8\,\pi\, T_{ab}$ and the entropy of any slice of LCH is same as that of the black-hole horizon in that theory given by,

$$S = \frac{1}{4}\int dA\, \sqrt{\gamma}\, \rho_B, \tag{2}$$

where $\rho_B/4$ is the entropy density of the black-hole horizon. For example, in general relativity ρ_B would be equal to 1. Let us now consider how this entropy changes along the null congruence k^a. The entropy change is

$$\delta S = \frac{1}{4}\int_0^{\lambda_f} d\lambda\, dA\, \sqrt{\gamma}\left(\rho_B\,\theta + \frac{d\rho_B}{d\lambda}\right). \tag{3}$$

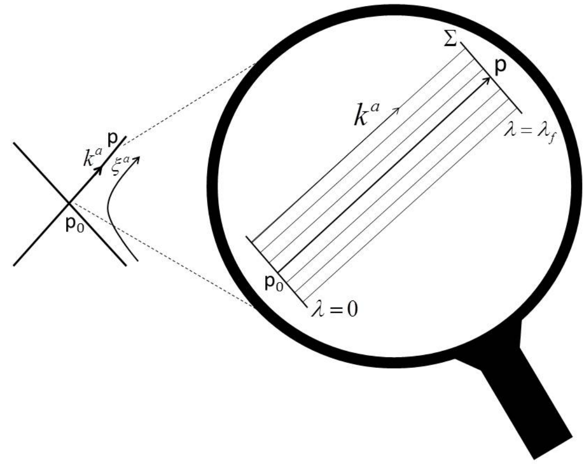

Figure 1: Local Causal Horizon (LCH) associated with the approximate boost Killing vector field ξ^a. p_0 is the bifurcation point where ξ^a vanishes. k^a is the affinely parametrized tangent to the null generators of the LCH with the affine parameter λ. p is the final equilibrium point where the expansion θ and shear σ of k^a vanishes. The affine parameter vanishes at p_0 and takes a value λ_f at p. On the generator connecting p_0 to p we have $\xi^a = \lambda k^a$.

[1] Our construction of the LCH is different from the one orginally given by Jacobson [6]. There the equilibrium was at the bifurcation point (where the Killing vector vanishes), while for us the equilibrium point (p) lies in the future of the bifurcation point (p_0). Both of these constructions are discussed in Ref. [13].

Next, we expand the entropy change in a Taylor series around the terminal cross section at $\lambda = \lambda_f$,

$$\delta S = \frac{1}{4}\int_0^{\lambda_f} d\lambda\, dA\,\sqrt{\gamma}\Big[\left(\rho_B\,\theta + \frac{d\rho_B}{d\lambda}\right)_{\lambda_f} + (\lambda-\lambda_f)\frac{d}{d\lambda}\left(\rho_B\,\theta + \frac{d\rho_B}{d\lambda}\right)_{\lambda_f} + \;\ldots\Big], \quad (4)$$

where the suffix λ_f indicates that the quantity in the bracket is evaluated at the final cross section $\lambda = \lambda_f$. Since $\int_0^{\lambda_f} d\lambda\,\lambda$ appearing in δQ in Eq. (1) is of the same order in λ_f as $\int_0^{\lambda_f} d\lambda\,(\lambda-\lambda_f)$, the second term of expansion in Eq. (4), for the validity of the Clausius relation we must have that the first term of expansion in Eq. (4) vanishes. By our construction of LCH, θ is zero at λ_f on the central generator. Hence a necessary condition for the validity of the Clausius relation for the LCH is that

$$\left(\frac{d\rho_B}{d\lambda}\right)_{\lambda_f} = 0. \quad (5)$$

In general relativity ρ_B is 1 and Eq. (5) is trivially satisfied. Hence, for general relativity we get from Eq. (4),

$$\begin{aligned}\delta S &= \frac{1}{4}\int_0^{\lambda_f} d\lambda\, dA\,\sqrt{\gamma}(\lambda-\lambda_f)\left(\frac{d\theta}{d\lambda}\right)_{\lambda_f} \quad (6)\\ &= \frac{1}{4}\int_0^{\lambda_f} d\lambda\, dA\,\sqrt{\gamma}(\lambda-\lambda_f)\left(-R_{ab}k^a k^b\right)_{\lambda_f},\end{aligned}$$

where in the second step we have used the Raychaudhari equation and the fact that $\theta(\lambda_f) = 0 = \sigma(\lambda_f)$. Now using the Einstein equation and doing the λ integral we see that Eq. (6) is equal to $\delta Q/T$, where δQ is as in Eq. (1) and T is the local Unruh Temperature, $T = 1/2\pi$. Hence it is easy to prove the Clausius relation for LCHs in general relativity. For any other theory of gravity, however, the situation is different. For example, consider a theory of gravity described by the Lagrangian which is a function of the Ricci scalar $f(R)$. The Wald entropy density associated with the black holes of such a theory is $\sim f'(R)$, and the validity of Eq. (5) requires that

$$\left(\frac{dR}{d\lambda}\right)_{\lambda_f} = 0.$$

But the spacetime is completely arbitrary and in general we can not set $dR/d\lambda = 0$ on the final cross section of the LCH.

It is worth pausing here to recall the setting of the physical-process version of the first law of black-hole thermodynamics: a black hole in an initial stationary state is perturbed by a small inflow of energy, finally it settles down to another stationary state and in the process there is some net change in its entropy [4, 15, 16]. Hence, for the physical processes involving black holes, the final cross-section is a part of a stationary spacetime, and therefore, all changes of the dynamical fields vanish in the asymptotic future. In particular, we can set $dR/d\lambda = 0$ on the final cross-section.

This is the crucial geometric difference between the case of black holes and that of the LCHs. The future boundary condition for black holes is physically motivated from the stability and cosmic censorship hypothesis [17]. There is no compelling reason to impose such a condition on the LCHs. This is the difficulty in imposing Eq. (5) necessary for deriving the Clausius relation for the LCHs.

It was suggested in Ref. [8] that one could impose a new equilibrium condition on the final cross-section of the LCH. Instead of requiring that $\theta = 0$ at p one could choose a patch Σ such that the first term of the expansion in Eq. (4) vanishes identically at p so that one doesn't need to impose Eq. (5) in addition to imposing $\theta = 0$ at p. Then one could derive a non-equilibrium version of the Clausius relation that contains an extra term which is interpreted as the entropy-production term. In Ref. [9], this entropy production term is shown to be a separate contribution to the heat flux coming from the additional scalar degree of freedom present in $f(R)$ gravity. Although such an interpretation offers an understanding of the entropy production terms for the specific case of $f(R)$ gravity, it is not clear how to generalize it for a broader class of theories. In fact, it is explained in Ref. [13] why such an approach may not work beyond the simple case of $f(R)$ gravity.

A new proposal for the entropy density.— In this paper we follow a different approach. We believe that there is no compelling reason why the expression for the entropy density of stationary black holes should agree with that of the LCHs. We therefore seek a new candidate for the entropy associated with the LCHs. But this expression for the entropy density can not be completely arbitrary. There is some minimal set of properties one expects from this entropy density: it should of course satisfy the Clausius relation; as in the black-hole case it is desirable to have $\theta = 0$ and $\sigma_{ab} = 0$ as the definition of the equilibrium slice of the LCH; for a stationary black hole the proposed entropy should agree with the entropy which satisfies the physical-process version of the first law for black holes. For example, for general relativity it should be proportional to the area of the horizon, for the Gauss-Bonnet gravity it should agree with the Jacobson-Myers expression [18].

With these guidelines in mind, our proposal is that the entropy density ρ of any slice Σ_λ of LCH, located between p and p_0 at an affine parameter $\lambda \in [0, \lambda_f]$, is given by

$$\rho = \rho_B + (\lambda_f - \lambda)\left(\frac{d\rho_B}{d\lambda}\right)_{\lambda_f} + \frac{(\lambda_f-\lambda)^2}{2}\left(\rho_B R_{ab}k^a k^b - \frac{d^2\rho_B}{d\lambda^2} - E_{ab}k^a k^b\right)_{\lambda_f}, \quad (7)$$

where $\rho_B/4$ represents the expression for the entropy density of the stationary black holes of the theory in ques-

tion. The entropy of the slice is given by

$$S = \frac{1}{4}\int_{\Sigma_\lambda} dA\sqrt{\gamma}\,\rho. \tag{8}$$

Note that the entropy density of the final equilibrium slice is equal to ρ_B, the entropy of the stationary black hole in the theory.

Let us now check if our expression of the entropy density leads to the correct thermodynamics of the LCH. The change in entropy as we evolve the system from the initial slice at $\lambda = 0$ to the final equilibrium slice at $\lambda = \lambda_f$ is given by Eq. (4), with ρ_B now replaced by ρ in Eq. (7), as

$$\delta S = \frac{1}{4}\int_0^{\lambda_f} d\lambda\, dA\,\sqrt{\gamma}\Bigg[\left(\rho\,\theta + \frac{d\rho}{d\lambda}\right)_{\lambda_f} + (\lambda-\lambda_f)\frac{d}{d\lambda}\left(\rho\,\theta + \frac{d\rho}{d\lambda}\right)_{\lambda_f} + \ \ldots\Bigg]. \tag{9}$$

Now, noticing that $\theta(\lambda_f) = 0$ because of our equilibrium condition, the first term in the big square brackets just gives $(d\rho/d\lambda)_{\lambda_f}$ which is easily seen to be equal to 0 from Eq. (7). The coefficient of $(\lambda-\lambda_f)$ can be calculated in a straightforward fashion using the Raychaudhari equation and it gives simply $(-E_{ab}k^ak^b)_{\lambda_f}$. In the limit that the initial slice gets very close to the final equilibrium slice we get,

$$\begin{aligned}\delta S &= \frac{1}{4}\int_{\Sigma_{\lambda_f}} dA\sqrt{\gamma}E_{ab}k^ak^b\int_0^{\lambda_f} d\lambda(\lambda_f-\lambda)\\ &= \frac{\lambda_f^2}{2}\,\frac{1}{4}\int_{\Sigma_{\lambda_f}} dA\sqrt{\gamma}E_{ab}k^ak^b.\end{aligned} \tag{10}$$

In the same limit we get, from Eq. (1) for the total heat flux through the null surface enclosed between $\lambda = 0$ and $\lambda = \lambda_f$,

$$\delta Q = \frac{\lambda_f^2}{2}\int_{\Sigma_{\lambda_f}} dA\,\sqrt{\gamma}\;T_{ab}k^ak^b. \tag{11}$$

Now using the local boost temperature $T = 1/2\pi$ and the equation of motion $E_{ab} = 8\pi T_{ab}$, we see from equations (10) and (11) that the Clausius relation $\delta S = \delta Q/T$ is satisfied at the LCH.

Discussion.— Few comments on our proposal for the entropy density associated with the LCHs are in order now. First of all, in the case of general relativity, $\rho_B = 1$ and $E_{ab}k^ak^b = R_{ab}k^ak^b$. Therefore, our entropy formula gives the usual area proportionality. Also, as we have already mentioned, on the final equilibrium slice our entropy density ρ agrees with ρ_B, the entropy density of the stationary black-hole horizon in the theory. One might think that the previous statement is a tautology because on the final equilibrium slice one has $\lambda = \lambda_f$, so from Eq. (7) the only surviving contribution to ρ is ρ_B. However, what is non-trivial is that for stationary black holes this should be true even for an arbitrary λ, provided that the expression represented by ρ_B satisfies the physical-process version of the first law of black-hole thermodynamics. In order to see this, we note that all the derivatives of ρ_B with respect to λ vanish on any stationary slice of the black-hole horizon. Furthermore, by stationarity θ, σ and $d\theta/d\lambda$ on a stationary slice are all zero too, so $R_{ab}k^ak^b$ is zero by the Raychaudhari equation. Finally, if ρ_B satisfies a physical process law, there can not be any flux of matter stress tensor across the stationary slice and hence $T_{ab}k^ak^b$ is zero which implies, by the equation of motion, that $E_{ab}k^ak^b$ is zero. Therefore, we see from Eq. (7) that for any stationary slice of the black-hole horizon, we have $\rho = \rho_B$. This immediately shows that even for an arbitrary cross-section of a non-stationary black hole, our entropy formula gives $\rho = \rho_B$ as long as the perturbed black hole finally settles down to a stationary configuration in the asymptotic future.

We emphasize that it is not an aesthetic property that we demand from a proposed entropy density of the LCHs but it is essential for the consistency of the proposal with the black-hole thermodynamics. The reason is simple: the point p could be on *any* slice of the final stationary horizon of a black-hole and ρ would still be required to reduce to ρ_B on this slice. From Eq. (7) this would be possible only if both the first-order and the second-order coefficient of $(\lambda_f - \lambda)$ were to be zero for a stationary black-hole horizon, which is guaranteed to happen if ρ_B satisfies the physical-process version of the first law for black holes. One could question the dependence of our ρ on the affine-parameter λ and on our choice of the null vector k^a. We should remember, however, in hydrodynamics there is no derivation of the expression for a non-equilibrium entropy. One introduces the viscosity coefficients by hand to make sure that the entropy production is positive in the system's approach to the equilibrium. The same is true here. We have introduced terms by hand which ensure that the Clausius relation is satisfied. Note that these terms are independent of the additive and the multiplicative ambiguity in the choice of the affine parameter λ. Furthermore, if the matter satisfies the null energy condition then validity of the Clausius relation immediately implies the validity of the second law of thermodynamics for quasi-stationary approach to equilibrium. Therefore, in our case entropy production is guaranteed to be positive if the matter satisfies the null energy condition. It would be nice to understand if the λ-dependent terms in our entropy density have an interpretation in terms of the hydrodynamic coefficients.

We should compare our entropy density to that proposed in Ref. [13]. The situation in our proposal is better than the "Noetheresque" proposal of Ref. [13]. There the entropy density depended upon the choice of the approximate boost Killing vector field ξ^a and one had to make sure that the Killing identity is satisfied to an appropriate order for the derivation to go through. In contrast, our entropy density depends upon the null generator k^a

of the LCH (relevant portion of) Σ which we chose to begin with such that its expansion and shear vanish at the equilibrium point p and we do not have to worry about the Killing identity at all. In both the cases though, the Clausius relation applies only to the complete patch of LCH between $\lambda = 0$ and λ_f. This is precisely analogous to what happens in the physical process version of the first-law for black holes.

How sensitive is our derivation of the Clausius relation to the choice of the equilibrium condition $\theta(\lambda_f) = 0$, $\sigma^{ab}(\lambda_f) = 0$? While $\theta(\lambda_f) = 0$ is essential for our derivation, we can easily relax the condition $\sigma(\lambda_f) = 0$. This results in an extra term in Clausius relation that has a natural interpretation as an internal entropy production term $\sim \sigma^{ab}\sigma_{ab}$ whose coefficient can be interpreted as the shear viscosity [8]. For black holes in general relativity and f(R) gravity these are precisely the terms which account for the tidal heating [9]. To the best of our knowledge the expression for the shear viscosity in general theories of gravity is not known.

Finally, we should mention that a physical interpretation of ρ has not emerged from our construction. We have a quantity that satisfies the Clausius relation for local causal horizons in all diffeomorphism invariant theories of gravity. But is there a derivation of this expression from some basic principles? As far as this question is concerned, the situation is better in the proposal of Ref. [13] because that construction is based upon Wald's Noether charge entropy of the black holes in the theory. It remains to be seen if a similar interpretation is possible for our proposal of the entropy density of the LCHs.

Acknowledgments.— We thank Stefano Liberati for stimulating discussions and detailed comments on an earlier draft of this paper. We would also like to thank Ted Jacobson for comments. AM thanks the Institute of Mathematical Sciences, India, for hosting him during the initial stages of this work. SS's research is partially supported by the IIT Gandhinagar internal project grant: IP/IITGN/PHY/SS/2013-001.

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